Three-Branched Linear Map as a Model for a Perturbed Oregonator

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In the present paper we examine the behavior of a 3-branched linear map which resulted from computations of the periodically perturbed Oregonator. It turns out that the linear map describes quite satisfactorily the period composition predicted by the chemical model and by the more elaborated nonlinear hyperbolic maps that were described in a previous publication. In particular, we have shown that in any interval of the control parameter, two periodic orbits, differing by one point, coexist with one being stable and the other unstable. This phenomenon is, to the best of our knowledge, described for the first time in the literature.

1. Introduction

One-dimensional return maps are very useful tool in the investigations of the dynamics of nonlinear systems. The most well known examples of such maps are the logistic map[1] and the sine or circle map.[2] Another, less known example, is the cusp map.[3] In the field of theoretical biology, Glass and Mackey[4] and Keener *et al*.[5] have used a variety of maps to investigate such problems as "integrate and fire" models. Swinney[6] and his collaborators have used logistic and sine maps for investigating various aspects of the BZ oscillations, period doubling and chaos formation. Two extrema maps were used by Ringland *et al*.[7] to explain the smooth transformation from U to Farey sequences observed in some cases. The maps can be constructed from the experimental data as well as from the numerical solutions of appropriate kinetic equations. Other types of maps are also possible, as the three-branched map to be described in the present paper.

Markman and Bar-Eli[8] studied a periodically forced Oregonator and found a parameter range near a saddle-node bifurcation with the following properties: (1) Patterns of large and small oscillations are found in a frequency locked intervals. (2) Between any two such intervals there is an interval with a concatenation of the two patterns. Following this paper we found a family of 1-D maps[9] which describes in a satisfactory way the calculated results.

The family of maps studied in ref 9 transforms the interval [0, 1] onto itself and consists of 3 branches, *i.e.*, it is discontinuous at two points (say k_2 and k_3). Its inverse also transforms the unit interval onto itself; it is a function and is discontinuous at two points (say l_2 and l_1). Every branch of the map is an increasing function of x and is determined by a hyperbolic curve. The family depends on the bifurcation parameter k_2 which defines the first point of the discontinuity of the map.

These nonlinear (hyperbolic) maps are difficult to study and we have chosen, therefore, to examine in more detail the behavior of linear maps which are a reasonable approximation to the maps described in ref 9.

The aim of the present paper is the description of discrete dynamics generated by linear maps with three branches and the comparison of the results with those of nonlinear maps and the kinetic differential equations studied in ref 8.

The paper is organized as follows: in section 2 we describe the properties of three-branched maps and define a family of linear maps. In section 3 we describe results and in section 4 the results are discussed, differences between our map and others are analyzed, and conclusions are presented.

2. Family of Three-Branched Maps

In nonlinear (hyperbolic) maps there may be two cases: (a) the first branch touches (and intersects) the diagonal before k_2 approaches k_3 or (b) the first branch does not touch the diagonal before k_2 approaches k_3.

It is obvious that in case a the first branch must be curved. Thus, the replacement of hyperbolic maps by linear ones is reasonable only in case b. We restrict our study to this latter case.

The three-branched linear map, called $f(x)$, is defined as follows:

$$f_1(x) = l_1 + \frac{(1 - l_1)}{k_2}x \quad \text{for} \quad 0 \le x \le k_2 \tag{1}$$

$$f_2(x) = l_2 + \frac{(l_1 - l_2)}{(k_3 - k_2)}(x - k_2) \quad \text{for} \quad k_2 < x < k_3 \tag{2}$$

$$f_3(x) = \frac{l_2}{(1 - k_3)}(x - k_3) \quad \text{for} \quad k_3 \le x \le 1 \tag{3}$$

With the conditions $0 < k_2 < k_3 < 1$ and $0 < l_2 < l_1 < 1$ in order that the three branches will always exist. Thus the map is defined on the interval $0 \le x \le 1$ and is discontinuous at $x = k_2$ and at $x = k_3$.

Since the map is invertible and linear, it is easy to write its inverse $g(x) = f^{-1}(x)$ as follows:

$$g_1(x) = \frac{1 - k_3}{l_2}x + k_3 \quad \text{for} \quad 0 \le x \le l_2 \tag{4}$$

$$g_2(x) = \frac{k_3 - k_2}{l_1 - l_2}(x - l_2) + k_2 \quad \text{for} \quad l_2 < x < l_1 \tag{5}$$

$$g_3(x) = \frac{k_2}{1 - l_1}(x - l_1) \quad \text{for} \quad l_1 \le x \le 1 \tag{6}$$

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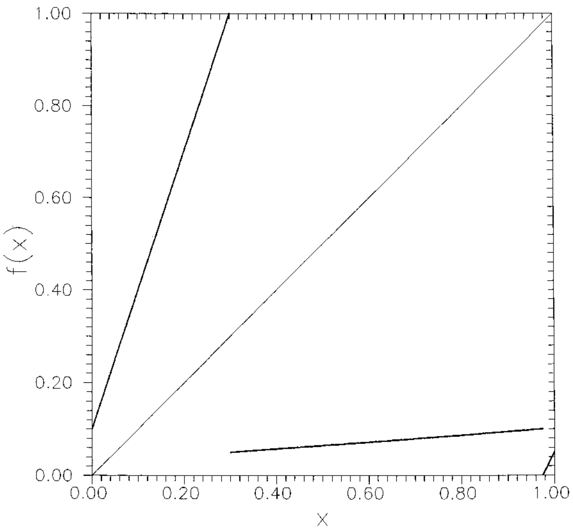

Figure 1. A plot of a typical 3 branch map (eqs 5−7) for the fixed parameters $l_1 = 0.1$, $l_2 = 0.05$, $k_3 = 0.975$ and the variable parameter $k_2 = 0.3$. The thick line is the map, while the thin line shows the diagonal.

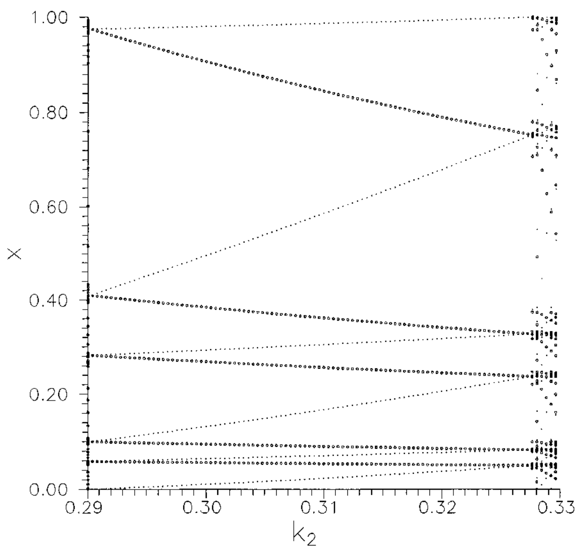

Figure 2. A plot of x vs $0.29 < k_2 < 0.33$, showing the stable orbits (thick lines) and the coexisting unstable ones (thin lines) which occur in this region. The pattern of the SO, is FSF_2S, while that of the UO, is $FFTF_2S$.

The inverse map transforms the unit interval onto itself and is discontinuous at $x = l_1$ and at $x = l_2$. This inverse map may prove useful when calculating the unstable orbits (see below).

The parameters used in our particular calculations are $l_1 = 0.1$, $l_2 = 0.05$, $k_3 = 0.975$, and k_2 is used as a control parameter. This linear map has, as can easily be seen, the general properties of the nonlinear maps described above, and can, therefore, serve as a good approximation for them in case b.

In Figure 1 a typical linear map for $k_2 = 0.3$ with its three branches is shown. On the left the first branch rises from $l_1 = 0.1$ at $x = 0$ to 1 at $x = k_2 = 0.3$; to the right of the diagonal, the second branch rises from $l_2 = 0.05$ at $k_2 + \delta$ to l_1 at $k_3 - \delta$; finally, the thirrd branch rises from 0 at k_3 to l_2 at 1. The figure is very similar to Figure 3 of ref 9 in which a three-branched hyperbolic map is shown, thus substantiating our assumption that linear maps may serve as a good simple way to study the more complicated nonlinear system.

A cycle of n points, $(x_1, x_2, ..., x_n)$ or an n periodic orbit, (whether stable or unstable) will exist if $f^n(x_i) = x_i$ ($i = 1, 2, ..., n$). The points will be termed F, S, or T if they belong to the first region (f_1), second region (f_2), or third region (f_3),

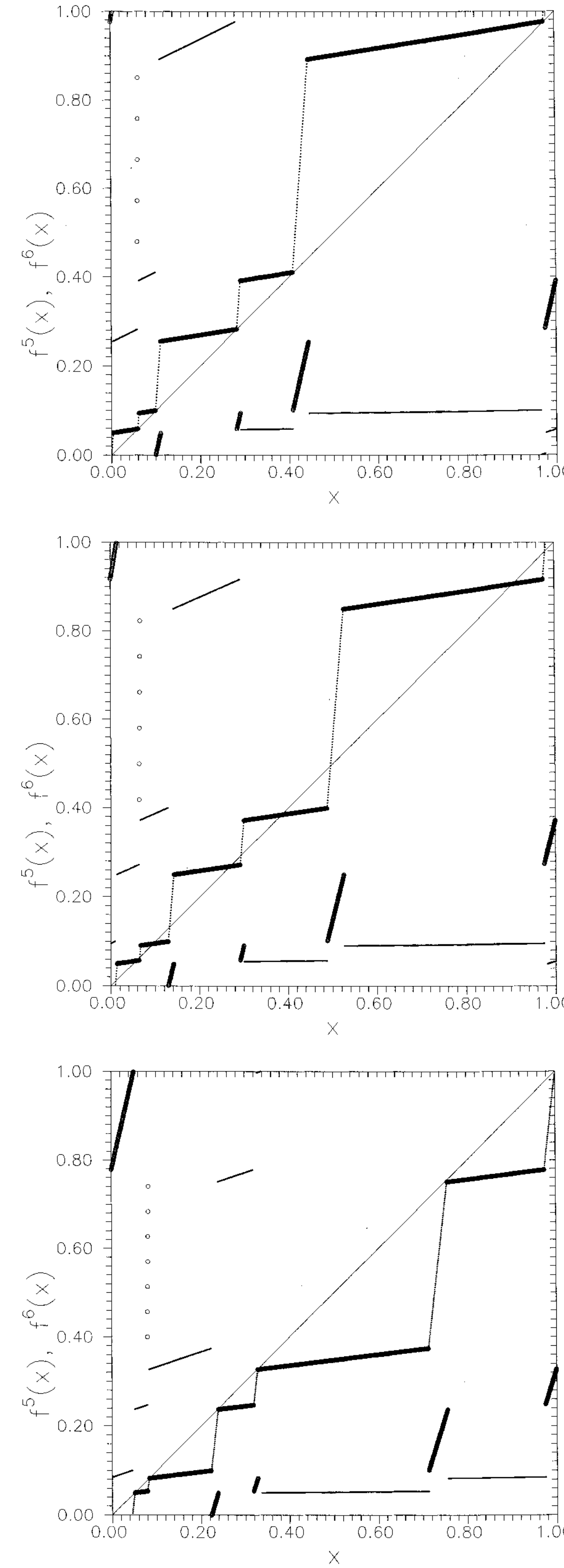

Figure 3. A series of plots showing the way the fifth (thick lines) and sixth (thin lines) iteration of the map change as k_2 move from $k_{2b} = 0.290\,241$ to $k_{2e} = 0.327\,53$. (The only significant “pieces” are those near the diagonal.) (a) $k_2 = k_{2b}$. The “staircase” of stable and unstable parts just touch the diagonal at the beginning of the region (b) $k_2 = 0.3$ The “staircase” is in the middle of the region. (c) $k_2 = k_{2e}$. The “staircase” just detach from the diagonal at the end of the region.

respectively. Thus a pattern will be a sequence of F, S, and T points. The stability of the orbit will be determined by the product of the slopes at the points x_i: the orbit will be stable (unstable) if the product is smaller (larger) than 1. Since the map is linear, it is easy to see that the slopes are given by

$$s_1 = \frac{1 - l_1}{k_2} = \frac{0.9}{k_2} \tag{7}$$

$$s_2 = \frac{l_1 - l_2}{k_3 - k_2} = \frac{0.05}{0.975 - k_2} \tag{8}$$

$$s_3 = \frac{l_2}{1 - k_3} = 2 \tag{9}$$

As the parameter k_2 changes, a variety of orbits and patterns will appear, only once, at some interval of k_2.

All periodic orbits appear as pairs of stable, SO, and unstable, UO, orbits, which differ by one in the number of points. Every pair SO and UO begins at k_{2b} where the appropriate "piece" of the nth and $(n + 1)$th iterate of k_3 touch the diagonal together. The interval for this pair of orbits ends at k_{2e} where the nth and $(n + 1)$th iterate detach the diagonal. The values of k_{2b} and k_{2e} are the solutions of the following equations:

$$f^{n+1}(k_3) = k_3 = 0.975 \tag{10}$$

$$f^{n+1}(k_{2e}) = k_{2e} \tag{11}$$

These equations are the $n + 1$ iterates of the discontinuities of the map. In what follows we show that at these points the nth and the $(n + 1)$th iterates coincide.

It is easy to see, from the definition of the map $f(x)$, that the following transformations are fulfilled:

$$\begin{aligned} 0 &\to l_1 \\ 0 + \delta &\to l_1 + \delta s_1 \\ k_2 - \delta &\to 1 - \delta s_1 \\ k_2 &\to 1 \\ k_2 + \delta &\to l_2 + \delta s_2 \\ k_3 - \delta &\to l_1 - \delta s_2 \\ k_3 &\to 0 \\ k_3 + \delta &\to 0 + \delta s_3 \\ 1 &\to l_2 \end{aligned} \tag{12}$$

The discontinuities at the break points k_2 and k_3 are clearly seen as $\delta \to 0$.

Suppose now that the point $k_3 - \delta$ belongs to an n periodic orbit at some k_2, slightly larger than k_{2b}. There will be a sequence of n points of the form (sequence I):

$$\overbrace{l_1 - \delta s_2, \ldots, k_3 - \delta,}^{n\ \text{points}} \quad l_1 - \delta s_2, \ldots, k_3 - \delta, \quad \ldots \tag{I}$$

As $\delta \to 0$, the first point will approach l_1, whereas the last point will approach k_3. At exactly $\delta = 0$ a new sequence (sequence II) with $n + 1$ points will be formed, namely,

$$\overbrace{k_3, 0, l_1, \ldots,}^{n+1\ \text{points}} \quad k_3, 0, l_1, \quad \ldots \tag{II}$$

which has an F and T points (with $x = 0$ and $x = k_3$, respectively), instead of the S (with $x = k_3 - \delta$) point.

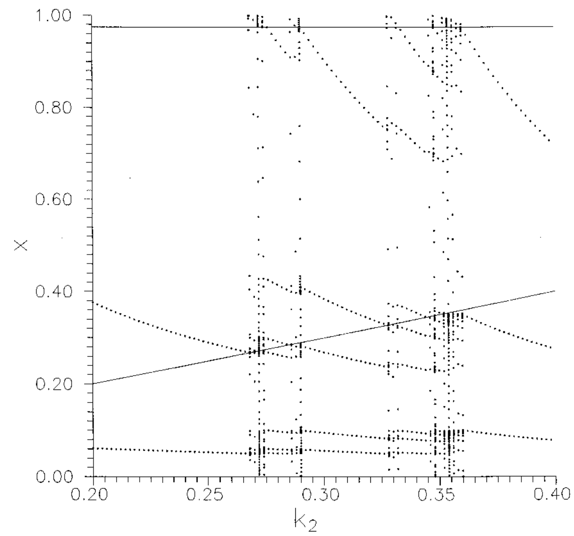

Figure 4. A typical histogram of $0.2 < k_2 < 0.4$. The values of k_2 and $k_3 = 0.975$ are shown as thin lines. Only the stable points are shown.

At the other end of the interval, when the point k_{2e} is approached, we obtain the following sequence of n points, namely (sequence III),

$$\overbrace{k_{2e} + \delta, l_2 + \delta s_2, \ldots,}^{n\ \text{points}} \quad k_{2e} + \delta, l_2 + \delta s_2, \quad \ldots \tag{III}$$

which has its limit in a the sequence IV of $n + 1$ points:

$$\overbrace{k_{2e}, 1, l_2, \ldots,}^{n+1\ \text{points}} \quad k_{2e}, 1, l_2, \ldots \tag{IV}$$

Again, sequence IV has an F and T ($x = k_{2e}$ and $x = 1$) points instead of an S ($x = k_{2e} + \delta$) point in sequence III.

By changing k_2 continuously from k_{2b} to k_{2e}, sequence I is changed smoothly to sequence III, while sequence II is changed to IV. Thus, at each value of k_2, in the interval $[k_{2b}, k_{2e}]$, two sequences (orbits, patterns) coexist: one with n points and the other with $n + 1$ points. The sequences or patterns will differ by one point where an S point is changed to an FT pair. One of the sequences will be stable and the other will be unstable (depending on the ratio s_1s_3/s_2 being larger or smaller than one).

The above description can be clearly seen in Figure 2 which shows a plot of n points of SO and $n + 1$ points of UO as a function of k_2 between 0.290 241 and 0.327 53. The plot, zigzag shaped, has the unstable portions (in thin lines) going up from left to right, while those of the stable portions (thick lines) are going down. At the extreme ends, *i.e.*, at k_{2b} and k_{2e}, the SO and UO join together, as described above.

In Figure 3 we show an example of the touching and detaching of the pair of patterns FSF_2S and $FFTF_2S$ by plotting the fifth and sixth iterates of the map at $k_{2b} = 0.290\,241$, $k_2 = 0.3$ at the middle of the interval and at $k_{2e} = 0.32753$. In these figures we see the fifth (thick lines) and sixth iterate (thin lines) form together a sort of a "staircase" where the SO form the flat parts (slope smaller than 1) and the UO form the steep "jumps" (slope larger than 1). Note that the nth iterate of the three-branched map has $2n + 1$ "pieces" thus the figure contains some more points than the "staircase" of interest. As will be seen below, there may be more than one solution to equations 10 and 11, *i.e.*, there may be more than one pattern with the same number of points, existing, of course, in different regions of k_2.

3. Results

In Figure 4 we see a typical histogram showing the dependence of the *stable* orbits on the parameter k_2 in the region

TABLE 1: Main Patterns of the Type FS_n

pattern (stable)	pattern (unstable)	k_{2b}	k_{2e}	$k_{2b} - k_{2e}$
S	FT	0.0	0.05	0.05
$FS_5(FT)_2$	$FS_4(FT)_3$	0.050 001 4	0.050 006 4	5e−6[a]
$FS_4(FT)_2$	$FS_3(FT)_3$	0.050 008 3	0.050 031	2.27e−5
FS_4FT	$FS_3(FT)_2$	0.050 042 9	0.050 129 2	8.63e−5
FS_3FT	$FS_2(FT)_2$	0.050 170 5	0.050 947 2	7.767e−4
FS_3	FS_2FT	0.051 491 8	0.052 520 1	1.0283e−3
FS_2	FSFT	0.053 997 3	0.081 809 9	2.78126e−2
FS	FFT	0.102 857 14	0.267 944	1.65086e−1

[a] Read as 5×10^{-6}.

[0.2, 0.4]. The lines k_2 and k_3 are also drawn in order to stress the various points F, S, or T. On the left the pattern FS is clearly seen, while on the right one sees the F_2S pattern. The composed FSF_2S pattern appears in the middle. Other compositions with more complicated patterns, occupying rather narrow regions, can be also seen in this interval. This behavior is typical for the three regions of k_2, which are above l_2, namely, between any two intervals belonging to the same region and having orbits with n and m points, there is an interval with $n + m$ points with the combined pattern. At times the combined orbit will have $n + m + 1$ points due to the introduction of the FT sequence instead of S. This point will be treated in more detail below.

We shall discuss our results in four regions in which, roughly speaking, the main stable patterns are S, which occurs at $0 < k_2 < l_2$, FS_n (with variations due to the exchange of Ss for FTs) which occurs between l_2 and $k_2 = l_1(1 - l_1)/(k_3 - l_1) = 0.1029$ where the pattern FS begins, F_nS, from the beginning of FS to the end of stable F_nS, and F_nT which ends near k_3.

3.1. Region I. In the region $0 < k_2 < l_2$, the function f_2 intersects the diagonal and $s_2 < 1$. Therefore the pattern S is stable all the way. However, the pattern FT will also exist in this region and will be unstable because $s_3s_1 > 1$ (see eqs 7−9).

In this simple case, we can calculate exactly the dependence of S and FT patterns on the parameter k_2, namely,

$$x_S = k_2 + \frac{(l_2 - k_2)(k_3 - k_2)}{(k_3 - k_2) - (l_1 - l_2)} \tag{13}$$

$$x_F = \frac{k_2 l_2 (k_3 - l_1)}{l_2(1 - l_1) - k_2(1 - k_3)} \tag{14}$$

$$x_T = l_1 + \frac{l_2(1 - l_1)(k_3 - l_1)}{l_2(1 - l_1) - k_2(1 - k_3)} \tag{15}$$

These values give, of course, $f(x_S) = x_S$ for the stable pattern, $f(x_F) = x_T$, and $f(x_T) = x_F$, for the unstable one.

3.2. Region II. Table 1 shows the main patterns in the region $l_2 < k_2 < l_1(1 - l_1)/(k_3 - l_1) = 0.1029$ together with the k_2 intervals in which they exist. Patterns of the previous and the following regions are included for completeness. The general form of the main patterns in each interval is $FS_{m-k}(FT)_k$ appearing with its unstable partner $FS_{m-k-1}(FT)_{k+1}$ in the same interval (where $m = 1,2, ...,$ and $k = 0, 1, ...$). Decreasing k_2, an S is added at each step, thus increasing m by 1 and keeping k constant. After some steps of increasing m, k (and not m) will increase by 1, *i.e.*, FT will be added, instead of S, to obtain a new stable pattern with its new unstable partner. Thus, as can be seen from Table 1, as k_2 decreases the patterns are changed by adding S, thereby increasing from FS to FS_2 to FS_3 (increasing m by 1). The next step should have been FS_4, but this pattern does not exist at all (eqs 10 and 11 have no solutions), and the pair FS_3FT and $FS_2(FT)_2$ appears instead. The change from adding an S to adding an FT occurs when the adding of the latter pattern causes the new pattern to become stable. Thus, by adding an FT to FS_3 we obtain an FS_3FT which is stable (slope smaller than 1), and again, adding FT to FS_4FT results in a stable $FS_4(FT)_2$, while FS_5FT does not exist.

As the pattern becomes more complicated, *i.e.*, when it includes more points, its range becomes smaller, as can be seen from the last column of the table.

As k_2 increases, the number of Ss decreases until we reach the next region with FS.

TABLE 2: Main Patterns of the F_nS Type

pattern (stable)	pattern (unstable)	k_{2b}	k_{2e}	$k_{2e} - k_{2b}$
FS	FFT	0.102 857 14	0.267 944	1.65086e−1
F_2S	F_2FT	0.36	0.472 216	1.12216e−1
F_3S	F_3FT	0.548 522	0.615 817	6.72950e−2
F_4S	F_4FT	0.672 068	0.713 194	4.11260e−2
F_5S	F_5FT	0.754 566	0.780 454	2.58880e−2
F_6S	F_6FT	0.811 621	0.828 255	1.66340e−2
F_7S	F_7FT	0.852 422	0.863 186	1.07240e−2
F_8S	F_8FT	0.882 454	0.889 348	6.89400e−3
F_9S	F_9FT	0.905 104	0.909 363	4.25900e−3
$F_{10}S$	$F_{10}FT$	0.922 541	0.924 956	2.41500e−3
$F_{11}S$	$F_{11}FT$	0.9362	0.937 296	1.09600e−3
$F_{12}S$	$F_{12}FT$	0.947 059	0.947 193	1.34000e−4

[a] Read as 1.65086×10^{-1}.

3.3. Region III. In the region $l_1(1 - l_1)/(k_3 - l_1) < k_2 < 0.95$ the slope s_2 is smaller than s_3, and the main sequence of the stable patterns is of the form F_nS. For these patterns eqs 10 and 11 can be simplified using function f_1 only,

$$f^{n+1}(k_3) = f^{n-1}f(f_3(k_3)) = f^{n-1}f_1(0) = f_1^{n-1}(l_1) = k_3 \tag{16}$$

for finding k_{2b} and

$$f^{n+1}(k_{2e}) = f^{n-1}f(f_1(k_{2e})) = f^{n-1}f_1(1) = f_1^{n-1}(l_2) = k_{2e} \tag{17}$$

for finding k_{2e}. The last equalities can be written explicitly as simple polynomials in $1/k_2$, and in the case of our constant parameters they are

$$0.1\sum_{i=0}^{n-1}\left(\frac{0.9}{k_{2b}}\right)^i = k_3 \tag{18}$$

$$0.1\sum_{i=0}^{n-2}\left(\frac{0.9}{k_{2e}}\right)^i + 0.05\left(\frac{0.9}{k_{2e}}\right)^{n-1} = k_{2e} \tag{19}$$

It is seen that the number of solutions to the above equation is limited since for $n = 17$ $k_{2b} = 0.973\,627$, but for $n = 18$, $k_{2b} = 0.977\,679$ which is greater than k_3 and therefore inapplicable.

Table 2 summarizes these results. We observe the adding of Fs, which is a typical phenomenon caused by the shifting of a branch (in our case the first) towards the diagonal.[3]

The range of existence of the patterns decreases with n as in the previous region. In fact, up to the point where k_{2b} becomes larger than k_{2e} (see next section), the range decreases as a power: a plot of ln(range) vs n is a straight line with a slope of −0.464.

3.4. Region IV. At $n = 14$ the value of k_{2b} becomes larger than that of k_{2e} (last column of Table 3), and at the same time the patterns change their stability, *i.e.*, $F_{n-1}FT$ becomes stable, while $F_{n-1}S$ becomes unstable, contrary to the previous cases. Table 3 shows the main patterns appearing in this range when F_nT patterns are stable. These patterns go up to $n = 17$, beyond

TABLE 3: Main Patterns of the F_nT Type

pattern (unstable)	pattern (stable)	k_{2b}	k_{2e}	$k_{2e} - k_{2b}$
$F_{13}S$	$F_{13}FT$	0.955 803	0.955 224	−5.7900e−4
$F_{14}S$	$F_{14}FT$	0.962 922	0.961 808	−1.1140e−3
$F_{15}S$	$F_{15}FT$	0.968 775	0.967 253	−1.5220e−3
$F_{16}S$	$F_{16}FT$	0.973 627	0.971 793	−1.8340e−3

[a] Read as -5.7900×10^{-4}.

which the values of k_{2b} and k_{2e} become larger than k_3, as stated earlier, and our map looses its meaning.

The change of stability of the patterns $F_{n-1}S$ and $F_{n-1}FT$ will occur when $s_2 = s_1 s_3$, which for our set of parameters, is at $k_2 = 0.948\,649$. As can be seen from Tables 2 and 3, this point is between the end of $F_{12}S$ and the beginning $F_{13}S$.

Table 3 shows also that the ranges of existence of the patterns F_nFT increase with n and k_2, in contrast to what is observed in Tables 1 and 2.

3.5. Composition Patterns. The regions described above show the existence of simple patterns of the types FS_n (and variations) and F_nS (and variations). As we have seen in Figure 4, two such simple patterns may combine to form a concatenated pattern. Thus, between the patterns FS and FS_2 there is a composed pattern $FSFS_2$, and between FS and F_2S there is FSF_2S. In Table 4 a variety of such composition patterns is shown. In this section, the behavior of these composition patterns will be described with the aid of the Farey arithmetic.[12]

We denote the number of F points in a stable periodic orbit by n_F and the sum of the number of S and T points by n_{ST}. In this way we can assign a Farey quotient to each orbit as

$$\frac{n_F}{n_F + n_{ST}} \tag{20}$$

The coexisting unstable orbit will have the same total number of S and T points (since the number of Ss decreases, while that of the Ts increases by 1) while the number of Fs increases by 1. The Farey quotient will be, therefore,

$$\frac{n_F + 1}{n_F + 1 + n_{ST}} \tag{21}$$

Two Farey quotients

$$\frac{n_F}{n_F + n_{ST}} \tag{22}$$

$$\frac{m_F}{m_F + m_{ST}} \tag{23}$$

are called neighbors if their cross multiplication equals 1, *i.e.*

$$n_F(m_F + m_{ST}) - m_F(n_F + n_{ST}) = n_F m_{ST} - m_F n_{ST} = 1 \tag{24}$$

(assuming that the n ratio is larger than that of the m one).

Two stable periodic orbits, which are Farey neighbors, may be combined to form a new stable orbit, with Farey quotient, which is the Farey sum of the n and m quotients, namely,

$$\frac{n_F}{n_F + n_{ST}} \oplus \frac{m_F}{mF + m_{ST}} = \frac{n_F + m_F}{n_F + n_{ST} + m_F + m_{ST}} \tag{25}$$

The combined orbit can be easily verified to be a neighbor of both its "parents". Thus in the examples given above FS and FS_2 with Farey quotients $^1/_2$ and $^1/_3$, respectively, combine to give $FSFS_2$ with Farey quotient $^2/_5$ which is a neighbor of its parents. Also FS and F_2S ($^1/_2$ and $^2/_3$) give FSF_2S ($^3/_5$). More examples are given in Table 4, with the Farey quotients shown in parentheses.

The entries in the table show that in some cases combination occurs between one stable orbit and the unstable partner of its neighbor. When such combination occurs, the cross multiplication of the corresponding quotients will be

$$\begin{aligned}(n_F + 1)(m_F + m_{ST}) - m_F(n_F + 1 + n_{ST}) = \\ n_F m_{ST} + m_{ST} - m_F n_{ST} = 1 + m_{ST}\end{aligned} \tag{26}$$

Thus, combinations are possible also between patterns that their cross multiplication differ by $1 + m_{ST}$, provided that the relevant combination exists, *i.e.*, that eqs 10 and 11 have a solution.

For instance, in the range $0.272\,335 < k_2 < 0.272\,553$, $(FS)_2FFTF_2S$ ($^6/_{10}$) is obtained from the stable pattern $(FS)_2F_2S$ ($^4/_7$) and the unstable pattern FFT ($^2/_3$). The cross multiplication is equal to $2 \times 10 - 3 \times 6 = 2 = 1 + m_{ST}$. [Combining the

TABLE 4: Combination Patterns Between F_2S, FS, FS_2, FS_3, and FS_3FT

pattern (stable)	pattern (unstable)	k_{2b}	k_{2e}	$k_{2b} - k_{2e}$
F_2S ($^2/_3$)	F_2FT ($^3/_4$)	0.36	0.472 216	1.122e−1
$FS(F_2S)_3$ ($^7/_{11}$)	$FFT(F_2S)_3$ ($^8/_{12}$)	0.348 246	0.351 341	3.095e−3
$FS(F_2S)_2$ ($^5/_8$)	$FFT(F_2S)_2$ ($^6/_9$)	0.333 221	0.344 651	1.143e−2
$FSF_2SFFT(F_2S)_2$ ($^9/_{14}$)	$FFTF_2SFFT(F_2S)_2$ ($^{10}/_{15}$)	0.329 865	0.331 040	1.175e−3
FSF_2S ($^3/_5$)	$FFTF_2S$ ($^4/_6$)	0.290 241	0.32 753	3.729e−2
$(FS)_2F_2S$ ($^4/_7$)	$FSFFTF_2S$ ($^5/_8$)	0.275 93	0.28 485	8.92e−3
$(FS)_2FFTF_2S$ ($^6/_{10}$)	$FS(FFT)_2F_2S$ ($^7/_{11}$)	0.272 335	0.272 553	2.18e−4
$(FS)_3FFTF_2S$ ($^7/_{12}$)	$(FS)_2(FFT)_2F_2S$ ($^8/_{13}$)	0.269 110	0.270 515	1.405e−3
$(FS)_4FFTF_2S$ ($^8/_{14}$)	$(FS)_3(FFT)_2F_2S$ ($^9/_{15}$)	0.268 337	0.268 859	5.22e−4
$(FS)_5FFTF_2S$ ($^9/_{16}$)	$(FS)_4(FFT)_2F_2S$ ($^{10}/_{17}$)	0.268 152	0.268 193	4.1e−5
$(FS)_5(FFT)_2F_2S$ ($^{11}/_{19}$)	$(FS)_4(FFT)_3F_2S$ ($^{12}/_{20}$)	0.268 009	0.268 073	6.4e−5
FS ($^1/_2$)	FFT ($^2/_3$)	0.102 857	0.267 944	1.651e−1
$(FS)_3FS_2$ ($^4/_9$)	$(FS)_3FSFT$ ($^5/_{10}$)	0.099 144 6	0.099 594 8	4.502e−4
$(FS)_2FS_2$ ($^3/_7$)	$(FS)_2FSFT$ ($^4/_8$)	0.094 725 4	0.097 590 5	2.8651e−3
$FSFS_2$ ($^2/_5$)	$FSFSFT$ ($^3/_6$)	0.083 880 3	0.093 056 9	9.1766e−3
$FS_2FSFTFS$ ($^4/_9$)	$(FSFT)_2FS$ ($^5/_{10}$)	0.082 328 3	0.083 333 6	1.005e−3
FS_2 ($^1/_3$)	$FSFT$ ($^2/_4$)	0.053 997 3	0.081 809 9	2.78126e−2
$(FS_2)_2FS_2FT$ ($^4/_{11}$)	$FS_2FS_2FTFSFT$ ($^5/_{12}$)	0.053 887 7	0.053 947 3	5.96e−5
FS_2FTFS_2 ($^3/_8$)	$FS_2FTFSFT$ ($^4/_9$)	0.053 063 6	0.053 841 5	7.779e−4
$(FS_2FT)_2FS_2$ ($^5/_{13}$)	$(FS_2FT)_2FSFT$ ($^6/_{14}$)	0.052 788 3	0.053 012 1	2.238e−4
FS_3 ($^1/_4$)	FS_2FT ($^2/_5$)	0.051 491 8	0.052 520 1	1.0283e−3
FS_2FTFS_3FT ($^4/_{11}$)	FS_2FTFS_2FTFT ($^5/_{12}$)	0.051 008 0	0.051 224 3	2.163e−4
FS_3FT ($^2/_6$)	$FS_2(FT)_2$ ($^3/_7$)	0.050 170 5	0.050 947 2	7.767e−4

[a] Read as 1.122×10^{-1}.

neighbor *FS* ($^1/_2$), one should have obtained $(FS)_3F_2S$ ($^5/_9$), but this pattern does not exist.] This behavior is very similar to what one sees in Table 1, where *FT*s are added from time to time instead of an *S*.

Composed patterns can be formed, of course, also between patterns from different regions: the unstable pattern $F_{12}FT$ (from region III) is combined with the stable $F_{13}FT$ pattern (region IV) to form the stable $F_{13}TF_{14}T$ and its unstable partner $F_{13}TF_{13}S$. This occurs between $k_{2b} = 0.951\,533$ and $k_{2e} = 0.951\,529$, *i.e.*, between regions III and IV. Note that $k_{2b} > k_{2e}$, since both are larger than $k_2 = 0.948\,649$, where the stability change occurs. Finding the appropriate compositions of Table 4 can only be calculated by solving eqs 10 and 11.

The general rule for forming a combination of a stable orbit with the partner of its neighbor can be formed as follows: assume that a pattern *IS* is added each time to the sequence *JS* (where *IS* and *JS* are neighbor sequences), thus obtaining *ISJS*, $(IS)_2JS$, ... The unstable partners will be *IFTJS*,*ISIFTJS*, ... Since the sequence *IS* is stable then at each addition the slopes of both partners will decrease. At a certain point, say the third addition, the slope of *ISISIFTJS* will become smaller than 1. At this point, the sequence *ISISIFTJS* will become stable, *ISISISJS* will not exist, and a new unstable partner will be formed, namely, *ISIFTIFTJS*.

In the example given above for $k_2 = 0.2724$, we obtain that the pattern $(FS)_3F_2S$ does not exist and the pattern $(FS)_2FFTF_2S$ and its unstable partner $FS(FFT)_2F_2S$ have the slopes 0.937 612 and 287.648, respectively, thus confirming the above rule.

4. Discussion

In many oscillatory chemical systems the oscillations are composed of series of small and large amplitude peaks.[8,10,11] Analysis of the next amplitude maps of the small amplitude oscillations obtained from the perturbed Oregonator[8,9] revealed that in some cases a three-branched map appears. These maps have the following properties: (a) each branch increases monotonically and (b) the maps are invertible, which means that their reciprocals are single valued. As the mathematics of three-branched maps is comparatively unknown, we have constructed and studied in a previous publication[9] a family such three-branched maps containing hyperbolas. A very good agreement with the calculated results of Markman and Bar-Eli[8] has been obtained. In particular, the composition rule of the obtained patterns follows roughly the Farey arithmetic[12] eq 25. The composition occurs between those patterns that are neighbors in the sense of eq 24. However, a closer look at the results shows that the simple rule of the period combination does not always follow the Farey arithmetic, but that at certain intervals there is an extra point on the combined orbit due to the existence of the two points *FT* instead of the single *S* point. In order to facilitate the investigation of such three-branched maps, we have studied in this work three-branched linear maps that approximate, quite adequately, the three hyperbolic branches. We have found, first of all, that all periodic orbits (patterns and sequences) appear in pairs, one stable SO and the other unstable UO. The pairs differ in one point: the UO have two points *FT* instead of an *S* of the SO. (for $k_2 > 0.95$ the stable partner will be the one with *FT*, while the unstable partner will have an *S* instead, as explained in the text above). The coexistence of two orbits, one stable and the other unstable that differ in one point, is a new phenomenon which is characteristic to the three-branched maps and has not been described hitherto in the literature. The existence of the unstable "partner" explains the discrepancies obtained in the Farey rule pattern composition. As orbits combine, the combination may occur between two SOs but also may occur between an SO and the "partner" UO (with an extra point) of the other "parent". In other words the combination may occur not only between neighbors in the sense of eq 24, but also between patterns whose Farey ratios have a cross multiplication as in eq 26. Thus the combination rule, which was found both in the perturbed Oregonator and in the nonlinear hyperbolic three-branched map, finds a simple explanation through this work. This breaking of the Farey rule has been observed in the perturbed Oregonator close to a saddle-node bifurcation, therefore, one can expect the same phenomenon in other systems near a saddle-node bifurcation that are periodically perturbed.

The quantitative results presented here are dependent on the particular fixed parameters (such as k_3, l_1, etc.) that were used. Thus for example the finite limit of *F* which appear in subsequences ...F_nS... ($n = 12$), or finite limit of number of *F* which appear in subsequences ...F_nT... ($n = 17$), will change if other parameters were used. However, the qualitative results as the coexistence of stable and unstable orbits, the extension of the Farey "neighbor" (eq 26) will remain the same as described and will be applicable to nonlinear systems.

The results of our previous paper,[9] together with those presented above, are amenable to direct experimental test for periodical perturbations of the BZ reaction that is qualitatively described by the Oregonator model, or other similar models. The behavior and ordering of small amplitude oscillations can be examined and compared to the theoretical predictions given here.

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